Universality in self-organized critical systems

1 Self-organized critical systems

The term self-organized criticality (SOC) has been used to mean a variety of different things. Generally there's an element of scale-freedom involved, but if one uses the term in all its generality, this is all there is to be said about SOC. I will therefore focus on a subset of SOC systems that are amenable to investigations with the mathematical tools of statistical mechanics. In the following I will use SOC as having the following meaning:

SOC models are generally defined on lattices, discrete in space and time. They are driven by adding a particle to a lattice site. When a lattice site reaches a threshold number of residing particles, it dissipates them instantaneously to its neighbours. These may in turn reach their particle thresholds and consequently topple. In this manner an avalanche can be started. The avalanche finishes when all sites harbour no more particles than maximally allowed by their thresholds. Subsequently, the model is driven again. Particles are conserved within the bulk and dissipated at the boundaries, implying diverging event sizes in the thermodynamic limit.

SOC is a dynamical phenomenon. It commonly ocurrs in nature in slowly driven, spatially extended systems with fast dissipation mechanisms. Examples are rainfall [4, 3], earthquakes and stick-slip friction. In such systems relaxation, that is, energy dissipation, takes place in distinct burst without characteristic scale. This is expressed in the distribution of probablities $p(s)$ of event sizes s

$$p(s) = as^{-\tau}\mathcal{G}\left(s/s_{\xi}\right), \qquad \text{with } s_{\xi} = bL^{D}. \tag{1}$$

Equation 1 is written in close analogy with the equations describing simple scaling in equilibrium critical phenomena. It is valid for event sizes $s > s_0$, where s_0 denotes a lower cutoff that is independent of system size. The constants a and b correspond to non-universal metric factors and $\mathcal{G}(x)$ is a scaling function, which is typically constant for small arguments $x \ll 1$ and falls off more rapidly than any polynomial for large arguments, $\forall n, \lim_{x\to\infty} x^n \mathcal{G}(x) = 0$. The exponent D is called the gap exponent.

The distribution of dynamical events is reminiscent of *e.g.* the cluster size distribution near the percolation threshold, where the scaling function contains another argument, the ratio of system size L and spatial bulk correlation length ξ. SOC systems, however, are believed to be critical by defintion, wherefore this ratio is always zero.

2 Approaching criticality or fixed at criticality?

This last statement needs further clarification. There exists no non-trivial SOC model that has been solved analytically, and the only information we possess pertains to finite systems and finite-size scaling.

What does it mean for such a system to be "critical by definition"? There are essentially two possible interpretations. Firstly, it could mean that values of the relevant parameters of the model are defined to be at their (trivial) critical values somewhere in the definition of the model. The second view is that the model parameters only approach their critical values as L diverges.

The latter view is held by the proponets of the absorbing-state interpretaion of SOC. They believe that SOC systems are driven by their dynamics towards the critical point of an absorbing-state phase transtion. Imagine a typical SOC system, conservative in the bulk and dissipative at the boundaries. If we close the boundaries, *e.g.* by making them periodic, and observe the system at different fixed densities ζ of particles, it will undergo a continuous phase transition at a critical density from a low-density phase where it eventually settles in an absorbing state (all local particle numbers below threshold) to a high-density phase where it never finds an absorbing state. The proposition is that under open-boundary dynamics, the system is slowly driven towards the critical density, ζ_c, where activity ensues – activity can be measured as the density of above-threshold sites, ρ_a, and constitutes the order parameter of the absorbing-state phase transition. Activity, in turn, leads to transport of particles and eventually to dissipation at the boundaries, thus reducing the density in the system. Hence, there exists a coupling between order parameter and tuning field of the type

$$\partial_t \zeta(\rho_a, t) = h(L) - \epsilon(L)\rho_a(\zeta, t). \tag{2}$$

This picture introduces the driving time scales $h(L)$ and the dissipation time scale $\epsilon(L)\rho_s$, whose ratio determines the distance from the critical point, ζ_c, since at stationarity

$$\rho_a(L) = h(L)/\epsilon(L). \tag{3}$$

Since $\rho_a(L)$ is the value of the order parameter at a given finite system size, this translates into a value for the tuning field, $\Delta\zeta(L)$, which is determined not only by the singular pick-up of the order parameter at criticality but also by finite-size effects.

$$\Delta\zeta(L) \propto L^{-1/\mu}, \tag{4}$$

where μ is an effective critical exponent that can be calculated from the L-dependence of $h(L)$ and $\epsilon(L)$ as well as the exponents β, describing the increase of the order parameter in the high-density phase and γ, describing the order-parameter fluctuations in the low-density phase [5].

As the ratio in Eq. (3) approaches zero in the thermodynamic limit, the system approaches its critical point. The rationale behind a system-size dependence in h and ϵ goes as follows: As the system size increases, the avalanches become larger and longer in any sensibly defined measure of time. Since driving only takes place after avalanches have stopped, the driving rate h must decrease with system size to avoid overlapping avanlanches. The dissipation time scale $\epsilon\rho_a$ must also decrease, as it gets harder for larger systems to dissipate energy (the density of (dissipating) boundary site decreases as L^{-1}). Thus a given activity produces less dissipation – ϵ decreases with L.

This view of SOC gains credibility as numerical simulations of SOC systems and their closed-boundary absorbing-state counter parts are observed to share critical exponents and even the critical density ζ_c [1].

It is obvious, however, that the observed scaling behaviour is dictated by the system-size dependence of $h(L)$ and $\epsilon(L)$, as these govern the approach to criticality. Observing critical exponents through finite size scaling in such a system is equivalent to measuring finite-size scaling exponents in an Ising model while changing not only the system size but also, systematically, the temperature. To deduce the correct scaling exponents, one has to take into account the temperature dynamics [5]. Exponents from absorbing-state phase transitions and SOC agree – it is unclear to date how an SOC system "picks the right $h(L)$ and $\epsilon(L)$" to achieve this.

Another consequence of an approach to a critical point with diverging system size rather than a fixed position in parameter space at the critical point is a change in the ratio of $L/\xi(L)$, where $\xi(L)$ is not the bulk correlation length but the observed correlation length in the finite system. Direct observations of this ratio in SOC models, however, suggest that it is constant [2]. This would be expected if the system were fixed in parameter space at the critical point.

3 Investigation strategy

The questions raised in this discussion are

- Is the absorbing-state interpretation of SOC more than descriptive?
- How does the system find the correct scaling of h and ϵ, and why does simple scaling work if $L/\xi(L)$ is neither constant nor vanishes?
- What is the general form of $L/\xi(L)$?
- Most importantly, what is the relation between universality in equilibrium critical phenomena and dynamical, non-equilibrium, critical phenomena?

To answer these questions I will conduct further studies of absorbing-state and SOC systems, comparing critical properties of models from both worlds. I will measure directly the correlation length $\xi(L)$ in SOC systems. This requires the development of methods to deal with the highly non-uniform nature of correlations due to boundary effects in finite systems. Finally, I will conduct a literature-review of alleged universal properties of non-equilibrium critical phenomena and attempt to link them to the origins of universality in equilibrium critical phenomena.

References

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